

PSEUDO KNOTS AND AN OBSTRUCTION TO COSMETIC CROSSINGS

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ABSTRACT. Pseudo links have two crossing types: classical crossings and indeterminate crossings. They were first introduced by Ryo Hanaki as a possible tool for analyzing images produced by electron microscopy of DNA. A normalized bracket polynomial is defined for pseudo links and then used to construct an obstruction to cosmetic crossings in classical links.

1. INTRODUCTION

The set of pseudo knots and links was first introduced by Hanaki Ryo [2], to study the type of diagrams produced by electron microscopy of DNA. In these images, the over-under crossing information is often blurred; this results in a diagram with classical crossings and crossings where the under-over crossing information is unknown. Based on this physical interpretation, Ryo developed a set of Reidemeister-like moves that are not dependent on crossing type. Subsequent work by Allison Henrich explored several invariants of pseudo knots [4], [3], [5].

In this paper, we recall the definition of pseudo knots and links. Then, a modification of the bracket polynomial is defined for pseudo links. Finally, the pseudo bracket is applied to produce an obstruction to cosmetic crossings. A classical crossing x in a knot diagram D is said to be cosmetic if D is equivalent to the knot diagram D' where D' is obtained by switching the crossing x from a positively signed crossing to a negatively signed crossing (or vice versa). X. S. Lin conjectured that cosmetic crossings do not exist (with limited exceptions such as Reidemeister I twists and nugatory crossings). This is problem 1.58 on Kirby's problem list [7].

A pseudo link diagram D is a decorated immersion of n oriented copies of S^1 with two types of crossings. A crossing is either a classical crossing with over-under markings or a pseudo-crossing that is marked by a solid square as shown in Figure 1.

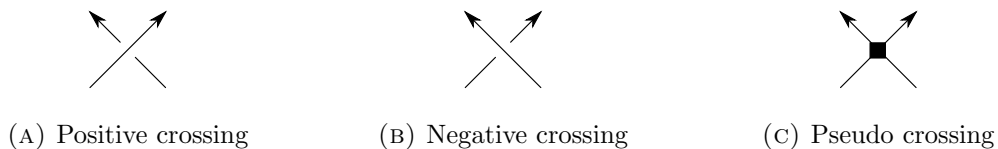


FIGURE 1. Crossing types

Classical crossings follow the usual sign conventions. For a positive crossing c , $\text{sgn}(c) = +1$ and for a negative crossing c , $\text{sgn}(c) = -1$.

Two pseudo link diagrams are equivalent if they are related by a sequence of Reidemeister moves (Figure 2) and Pseudo moves (Figure 3). A pseudo link is an equivalence class of pseudo link diagrams.

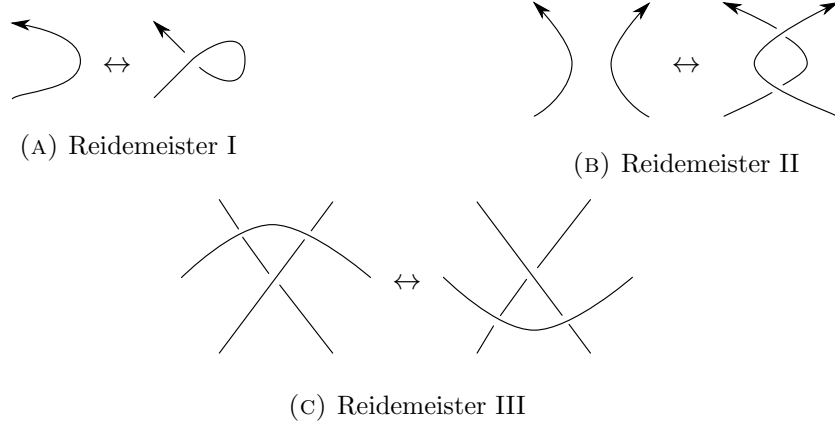


FIGURE 2. Reidemeister moves

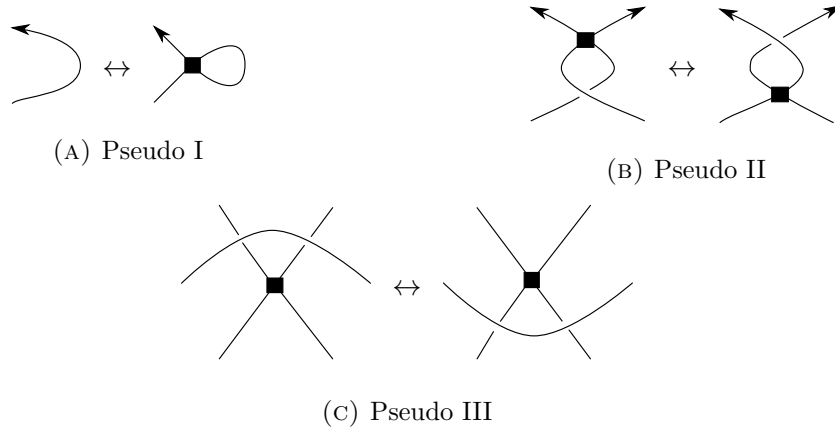


FIGURE 3. Pseudo moves

2. THE PSEUDO BRACKET POLYNOMIAL

Let D be an oriented pseudo link. The pseudo bracket polynomial is defined by a skein relation, building on the definition of the Kauffman bracket polynomial [6]. We expand a

positive crossing as:

$$(1) \quad \left\langle \begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right\rangle = A \left\langle \begin{array}{c} \curvearrowright \curvearrowleft \end{array} \right\rangle + A^{-1} \left\langle \begin{array}{c} \smile \frown \end{array} \right\rangle.$$

A negative crossing is expanded as:

$$(2) \quad \left\langle \begin{array}{c} \nwarrow \nearrow \\ \nearrow \nwarrow \end{array} \right\rangle = A \left\langle \begin{array}{c} \curvearrowleft \curvearrowright \end{array} \right\rangle + A^{-1} \left\langle \begin{array}{c} \smile \frown \end{array} \right\rangle.$$

Pseudo crossings are expanded as

$$(3) \quad \left\langle \begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right\rangle = V \left\langle \begin{array}{c} \curvearrowright \curvearrowleft \end{array} \right\rangle + H \left\langle \begin{array}{c} \smile \frown \end{array} \right\rangle$$

where $H = 1 - Vd$.

Let U denote the unknot and let $d = -A^2 - A^{-2}$. Expanding all crossings in a pseudo diagram results in a collection of simple closed curves. We evaluate a bracket containing an unlinked, simple closed curve using the simplifications:

$$(4) \quad \langle U \rangle = 1 \text{ and } \langle U \cup K \rangle = d \langle K \rangle.$$

The set of classical crossings in a link K is denoted as $C(K)$. The writhe of K is defined as

$$(5) \quad w(K) = \sum_{c \in C(K)} \text{sgn}(c).$$

Then, the normalized pseudo bracket of a pseudo link K is

$$(6) \quad P_K(A, V) = (-A^{-3})^{w(K)} \langle K \rangle.$$

Theorem 1. *For all pseudo links K , the pseudo bracket is invariant under Reidemeister moves II and III and the pseudo moves.*

Proof. Invariance under the Reidemeister moves is immediate. The skein relation on a classical link diagram gives the Kauffman bracket polynomial [6]. For a classical link K , $P_K(A, V) = f_K(A)$.

We show that the bracket is invariant under the Pseudo moves. We begin with the Pseudo III move.

$$\begin{aligned} \left\langle \begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right\rangle &= V \left\langle \begin{array}{c} \curvearrowright \curvearrowleft \end{array} \right\rangle + H \left\langle \begin{array}{c} \smile \frown \end{array} \right\rangle \\ &= V \left\langle \begin{array}{c} \curvearrowleft \curvearrowright \end{array} \right\rangle + H \left\langle \begin{array}{c} \smile \frown \end{array} \right\rangle \\ &= \left\langle \begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right\rangle. \end{aligned}$$

Next, the Pseudo II move.

$$\begin{aligned}
 \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \end{array} \right\rangle &= V \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right\rangle + H \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \right\rangle \\
 &= V \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right\rangle + H \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \right\rangle \\
 &= \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \end{array} \right\rangle.
 \end{aligned}$$

In the Pseudo I move,

$$\begin{aligned}
 \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \end{array} \right\rangle &= V \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right\rangle + H \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \right\rangle \\
 &= (Vd + H) \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right\rangle.
 \end{aligned}$$

Then $1 = Vd + H$ or $H = 1 - Vd$. □

Corollary 2. *For all pseudo links K , $P_K(A, V)$ is invariant under the Pseudo moves and the Reidemeister moves.*

Example 2.1. The pseudo bracket is applied to a trefoil with one pseudo crossing. This pseudo diagram is denoted as PT .

$$\begin{aligned}
 \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \end{array} \right\rangle &= A^2 \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \end{array} \right\rangle + 2 \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \end{array} \right\rangle + A^{-2} \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \end{array} \right\rangle \\
 &= A^2 \left(V \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right\rangle + H \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \right\rangle \right) + 2V \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right\rangle \\
 &\quad + 2H \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right\rangle + A^{-2} \left(V \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right\rangle + H \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} \right\rangle \right) \\
 &= -A^{-8}V + A^{-6} - A^4V
 \end{aligned}$$

Then, $P_{PT}(A, V) = A^{-12} + VA^{-14} - VA^{-2}$.

3. AN OBSTRUCTION

Let D_+ be a classical knot diagram with a selected positive crossing. The classical knot diagram D_- is obtained from D_+ by changing the selected crossing to a negative crossing. The pseudo diagram D_\blacksquare is obtained from D_+ by changing the selected crossing to a pseudo crossing. Suppose that the selected crossing is cosmetic and $D_+ \sim D_-$. Then we obtain the following theorem.

Theorem 3. *For all knot diagrams D_+ with a cosmetic crossing c , $\langle D_\blacksquare \rangle$ divides $\langle D_+ \rangle$ and $\langle D_- \rangle$.*

Proof. Suppose $D_+ \sim D_-$ and that D_+ and D_- are related by a single crossing change. Let $w(D_+) = w + 1$ then $w - 1 = w(D_-)$. Let K be a diagram equivalent to D_+ with writhe w .

We use G to denote $\langle K \rangle$. The normalized f -polynomials of K , D_+ , and D_- are equivalent:

$$(7) \quad G(-A^{-3})^w = \langle D_+ \rangle (-A^{-3})^{w+1} \quad G(-A^{-3})^w = \langle D_- \rangle (-A^{-3})^{w-1}.$$

Reducing Equation 7,

$$G = \langle D_+ \rangle (-A^{-3}) \quad G = \langle D_- \rangle (-A^3).$$

We conclude that

$$(8) \quad \langle D_+ \rangle = G(-A^3) \quad \langle D_- \rangle = G(-A^{-3}).$$

Partially expand the diagrams D_+ and D_- at the selected crossing.

$$(9) \quad \langle D_+ \rangle = A\langle K_v \rangle + A^{-1}\langle K_H \rangle,$$

$$(10) \quad \langle D_- \rangle = A^{-1}\langle K_v \rangle + A\langle K_H \rangle.$$

Then substitute Equation 8 into Equations 9 and 10.

$$(11) \quad G(-A^3) = A\langle K_v \rangle + A^{-1}\langle K_H \rangle,$$

$$(12) \quad G(-A^{-3}) = A^{-1}\langle K_v \rangle + A\langle K_H \rangle.$$

Multiplying through Equations 11 and 12:

$$G(-A^2) = \langle K_v \rangle + A^{-2}\langle K_H \rangle,$$

$$G(-A^{-2}) = \langle K_v \rangle + A^2\langle K_H \rangle.$$

Eliminate K_V from the system of equations:

$$(13) \quad G(-A^2 + A^{-2}) = (A^{-2} - A^2)\langle K_H \rangle.$$

Reducing Equation 13, we obtain

$$(14) \quad G = \langle K_H \rangle.$$

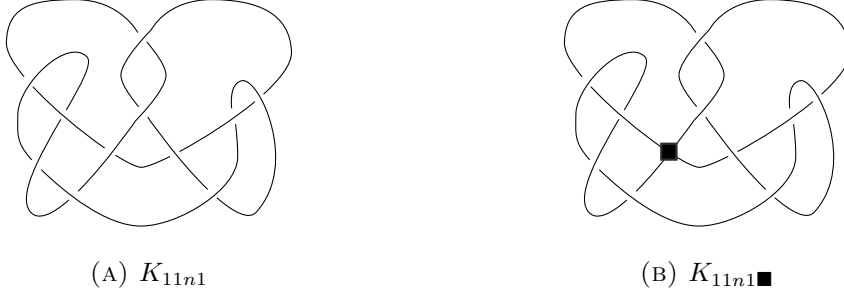


FIGURE 4. Example 2

Using the fact that $\langle D_+ \rangle = G(-A^3)$ and Equation 14,

$$\begin{aligned} G(-A^3) &= A\langle K_V \rangle + A^{-1}\langle K_H \rangle \\ G(-A^3) &= A\langle K_V \rangle + A^{-1}G \\ G(-A^3 - A^{-1}) &= A\langle K_V \rangle \\ Gd &= \langle K_V \rangle. \end{aligned}$$

As a result,

$$\langle K_V \rangle = dG \text{ and } \langle K_H \rangle = G.$$

Apply the result to the expansion of D_\blacksquare :

$$\begin{aligned} \langle D_\blacksquare \rangle &= V\langle K_V \rangle + (1 - Vd)\langle K_H \rangle \\ &= VdG + (1 - Vd)G \\ &= G. \end{aligned}$$

Then $\langle D_\blacksquare \rangle$ divides $\langle D_+ \rangle$ and $\langle D_- \rangle$. □

We obtain the following corollary.

Corollary 4. *For all knot diagrams D with a cosmetic crossing c , $\langle D_\blacksquare \rangle$ has no summands with a power of V .*

Example 3.1. The obstruction is demonstrated using the trefoil knot. By symmetry, none of the crossings are cosmetic.

$$\begin{aligned} \left\langle \begin{array}{c} \text{Trefoil knot with a crossing marked by a black square} \end{array} \right\rangle &= -A^{-8}V + A^{-6} - A^4V. \\ \left\langle \begin{array}{c} \text{Trefoil knot with a crossing marked by a black square} \end{array} \right\rangle &= A^{-7} - A^{-3} - A^5. \end{aligned}$$

Example 3.2. We consider the first non-alternating classical knot, K_{11n1} [1], shown in Figure 4a. In Figure 4b, we select a crossing to construct $K_{11n1}\blacksquare$.

$$(15) \quad \langle K_{11n1} \rangle = - \frac{(1 + 2A^8 - A^{12} + A^{28} - A^{32} + A^{36} - A^{40})}{(A^{17}(1 + A^4))}$$

$$(16) \quad \begin{aligned} \langle K_{11n1\blacksquare} \rangle = & A^{-24}(A^2 - 3A^6 + 5A^{10} - 7A^{14} + 9A^{18} - 9A^{22} + 8A^{26} - 6A^{30} \\ & + 4A^{34} - 2A^{38} + A^{42} + V - 3A^4V + 4A^8V - 6A^{12}V \\ & + 6A^{16}V - 5A^{20}V + 4A^{24}V - 2A^{28}V + A^{36}V - A^{40}V + A^{44}V) \end{aligned}$$

This calculation determines that the selected crossing in K_{11n1} is not cosmetic.

4. CONCLUSION

The pseudo bracket polynomial is not only an invariant of psuedo knots, but also provides a computable obstruction to cosmetic crossings.

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